$$ax+by+cz=d$$

$$2x+y+z=10$$

$$\underbrace{\overset{V_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}},\ \overset{V_2}{\begin{bmatrix}3\\0\\1\end{bmatrix}},\ \overset{V_3}{\begin{bmatrix}4\\2\\4\end{bmatrix}}}_{\text{وابسته خطی اند}}$$

$$\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$$

$$\begin{bmatrix}3\\2\end{bmatrix}$$

$$V_1+V_2=V_3$$

$V_1+V_2-V_3=0 \Rightarrow$ وابسته خطی اند

$V=[V_1, V_2, V_3] \rightarrow \det(V)=0 \Rightarrow$ سطرها {یا ستون‌ها} وابسته خطی اند

$$\sum_{j=-\infty}^{\infty} \longrightarrow \sum_{j=-\infty}^{-1} \quad j=0 \quad \sum_{j=1}^{\infty}$$

$$\sum_{n=1}^{\infty} a_n \longrightarrow \sum_{n=-1}^{-\infty} a_{-n} = \sum_{n=-\infty}^{-1} a_{-n}$$

$$n \to -n$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{a_n \cos nx + b_n \sin nx}_{c_n e^{inx} + k_n e^{-inx}}$$

$$c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} k_n e^{-inx}$$

$$n \to -n$$

$$\sum_{n=-1}^{-\infty} k_{-n} e^{inx}$$

$$c_{-n}$$

$$= \sum_{j=-\infty}^{\infty} c_j e^{inx}$$

$$(v, Lu) = (L^* v, u) \longrightarrow$$ L^* عملگر الحاقی L

adjoint

$$L = L^*$$ self_adjoint $\longrightarrow$ خود الحاقی

$$Lx = Ax \longrightarrow n \times 1 \Longrightarrow L^* = (\bar{A})^T$$

یک ماتریس n×n

$$A^* = (\bar{A})^T$$

$$H = (\bar{H})^T \longrightarrow$$ هرمیتی

Hermitian

مقادیر ویژه ماتریس هرمیتی حقیقی اند.

بردارهای ویژه متعامدند.

$$-y'' = \lambda y, \qquad \begin{cases} y(0) = y(2\pi) \\ y'(0) = y'(2\pi) \end{cases}$$

$$Ly = -y'' = -\frac{d^2}{dx^2}y \Rightarrow L = -\frac{d^2}{dx^2}$$

$Ly = \lambda y \longrightarrow$ مسئله مقدار ویژه

$L^* = ?$: می‌خواهیم عملگر الحاقی L را بدست آوریم

$$(v, Lu) = (v, -u'') = \int_0^{2\pi} v\,(\overline{-u''})\,dx$$

$$= \underbrace{v(-\bar{u}')\Big|_0^{2\pi}}_{0} - \int_0^{2\pi} v'\,(\overline{-u'})\,dx = (v', u')$$

$$= \underbrace{v'\bar{u}\Big|_0^{2\pi}}_{0} - (v'', u) = (-v'', u) = (Lv, u)$$

$L = L^* \Rightarrow$ L خودالحاقی

$Ly = \lambda y$ $\quad L = -\dfrac{d^2}{dx^2} \longrightarrow$ نعود الى ق

$y \neq 0 \rightarrow$ λ نعوض بصفر

$-y'' = \lambda y \Rightarrow y'' + \lambda y = 0$

قيم ذاتية

$\lambda = 0$

$-y'' = 0$

$y'' = 0$

$y(x) = Ax + B$

$y(0) = y(2\pi)$

$\Downarrow$

$B = A(2\pi) + B$

$\Downarrow A = 0$

$y' = A$

B ولكن

$B \neq 0 \rightarrow B = 1 \Rightarrow y(x) = 1 = \phi_0(x)$

$\lambda_0 = 0, \quad \lambda_n = n^2 \rightarrow n \in \mathbb{Z}^+$

$\phi_0(x) = 1$

$$\left\{ \begin{array}{l} \phi^{(1)}(x) = \cos nx \\ \phi^{(2)}(x) = \sin nx \end{array} \right.$$

متعامدين .

توابع ذاتية

$$L[y] = P_n \frac{d^n}{dx^n} y + P_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + \cdots + P_1 \frac{d}{dx} y + P_0 y$$

عملگر دیفرانسیل مرتبه n (خطی)

$$L[\alpha u + \beta v] = \alpha L u + \beta L v$$

$$L[y] = P_2 y'' + P_1 y' + P_0 y \leftarrow$$ عملگر دیفرانسیل مرتبه دوم (خطی)

$$L^*[y] = (-1)^n \frac{d^n}{dx^n}(\bar{P}_n y) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(\bar{P}_{n-1} y) + \cdots + (-1)\frac{d}{dx}(\bar{P}_1 y) + \bar{P}_0 y$$

$P_i \leftarrow$ به اندازه کافی مشتق پذیر باشند

$$\bar{v} L[u] - u \overline{L^*[v]} = \frac{d}{dx} B[u,v]$$ اتحاد لاگرانژ

$$B[u,v] = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j \geq 0,\ k \geq 0}} (-1)^j u^{(k)} (P_m \bar{v})^{(j)}$$

$$\int_a^b \left(\bar{v}\, L[u] - u\, \overline{L^*[v]}\right) dx = B[u,v]\Big|_a^b$$

$$= B[u,v]\Big|_{x=b} - B[u,v]\Big|_{x=a}$$

اتحاد گرین

Green

مثال.
$$L[y] = x^2 y'' + 2xy' + 3y$$

$$L^*[y] = (-1)^2 (x^2 y)'' + (-1)(2xy)' + 3y$$

$$= 2y + 4xy' + x^2 y'' - 2y - 2xy' + 3y$$

$$= x^2 y'' + 2xy' + 3y$$

$$L = L^*$$

$$L[y] = P_2 y'' + P_1 y' + P_0 y$$

P_2 ، P_1 ، P_0 توابع حقیقی

$$L^*[y] = (P_2 y)'' - (P_1 y)' + P_0 y$$

$$= P_2'' y + 2P_2' y' + P_2 y'' - P_1' y - P_1 y' + P_0 y$$

$$= P_2 y'' + (2P_2' - P_1) y' + (-P_1' + P_0 + P_2'') y$$

$$\begin{cases} P_1 = 2P_2' - P_1 \\ P_0 = -P_1' + P_0 + P_2'' \end{cases} \Longrightarrow \begin{cases} P_2' = P_1 \\ P_2'' = P_1' \end{cases}$$

$$L[y] = P_2 y'' + P_2' y' + P_0 y$$

$$= \frac{d}{dx}(P_2 y') + P_0 y$$

$$L[y] = y'' + p_1 y' + p_0 y = f(x)$$

$$e^{P(x)} = e^{\int p_1(x)\,dx}$$

$$e^{P(x)} \left(y'' + p_1 y' + p_0 y \right) = e^{P(x)} f(x)$$

$$\frac{d}{dx}\left[e^{P(x)} y' \right] + e^{P(x)} p_0 y = e^{P(x)} f(x)$$

Example. $$y'' + \frac{1}{x} y' + y = 0$$

$$e^{P(x)} = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x \qquad x$$

$$x y'' + y' + x y = 0 \implies \frac{d}{dx}(x y') + x y = 0$$

Result. If $L = L^*$ then the linear operator L is self-adjoint.

second order self-adjoint operators have the form

$$L[y] = \frac{d}{dx}[p y'] + q y.$$

Any differential Equation of the form

$$L[y] = y'' + p_1 y' + p_0 y = f(x)$$

j times continuously differentiable (real) functions

can be written as a self-adjoint equation.

$$e^{P(x)} = e^{\int p_1(x)\,dx}$$

Example. $\quad -y'' = \lambda y \ , \quad \underbrace{y(0) = y(\pi) = 0}$

$$\Downarrow$$

$$Ly = -y'' \qquad (u, Lv) = (L^* u, v)$$

$$\int_0^{\pi} \qquad L^* \to L$$

$$v(0) = v(\pi) = 0$$

$$u(0) = u(\pi) = 0$$

$$-y'' = \lambda y$$

$$\lambda = 0 \Rightarrow -y'' = 0 \Rightarrow y'' = 0$$

$$\Downarrow$$

$$\left.\begin{array}{l} y(x) = Ax + B \\ y(0) = 0, \ y(\pi) = 0 \end{array}\right\} \Rightarrow \underbrace{A = 0, \ B = 0}_{\Downarrow \ y = 0}$$

$\lambda = 0$ مقدار ویژه نیست. $\Longleftarrow \ y = 0$

$\lambda \neq 0 \qquad -y'' = \lambda y \qquad m^2 + \lambda = 0 \Rightarrow m = \pm\sqrt{\lambda}\, i$

$\rightarrow y(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$y(0) = 0 \;, \quad y(\pi) = 0$

$\Downarrow \qquad\qquad \Downarrow$

$A = 0 \qquad\qquad B\sin(\sqrt{\lambda}\,\pi) = 0$

$\Downarrow \qquad\qquad \Downarrow \qquad \Downarrow$

$\neq 0$

$\sin(\sqrt{\lambda}\,\pi) = 0 \Longrightarrow \sqrt{\lambda}\,\pi = n\pi \Longrightarrow \sqrt{\lambda} = n$

$\lambda_n = n^2 \,, \quad n \in \mathbb{N}$

$\lambda_n = n^2 \qquad \phi_n(x) = \sin(nx)$

مقادیر ویژه   توابع ویژه

$n = 1, 2, \ldots$